

# 2015 年北京大学化学体验营数学试题

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1. 设  $A, B, C, D, X$  为圆周上依次排列的五个点, 已知  $\angle AXB = \angle BXC = \angle CXD$ ,  $AX = a$ ,  $BX = b$ ,  $CX = c$ , 求  $DX$  的长.

**解析** 设  $AB = BC = CD = m$ ,  $AC = BD = n$ ,  $DX = d$ , 则由  $A, B, C, X$  四点共圆, 由托勒密定理可得

$$am + cm = bn,$$

由  $B, C, D, X$  四点共圆, 由托勒密定理可得

$$bm + dm = cn,$$

两式相除即得  $d = \frac{c^2 + ac - b^2}{b}$ .

2. 集合  $M = \{1, 2, \dots, 99\}$ , 集合  $A$  是集合  $M$  的子集,  $A$  中的元素个数为偶数, 且  $A$  中元素之和为奇数, 求符合要求的集合  $A$  的个数.

**解析** 易知集合  $A$  中奇元素的个数为奇数个, 偶元素的个数也为奇数个. 故符合要求的集合  $A$  的个数为

$$(C_{50}^1 + C_{50}^3 + \dots + C_{50}^{49}) (C_{49}^1 + C_{49}^3 + \dots + C_{49}^{49}) = 2^{97}.$$

3. 求证:

$$(1) \cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11} = -\frac{1}{2};$$

$$(2) \tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}.$$

**解析** (1)

$$\begin{aligned} & \cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11} \\ &= \frac{2 \sin \frac{\pi}{11} \left( \cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11} \right)}{2 \sin \frac{\pi}{11}}, \end{aligned}$$

对上式中的分子应用积化和差公式, 可得上式等于

$$\frac{\sin \pi - \sin \frac{\pi}{11}}{2 \sin \frac{\pi}{11}} = -\frac{1}{2}.$$

(2) 因为

$$\begin{aligned}
 & \left( \sin \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} \cos \frac{3\pi}{11} \right)^2 - 11 \cos^2 \frac{3\pi}{11} \\
 = & \left( \sin \frac{3\pi}{11} + 2 \sin \frac{5\pi}{11} - 2 \sin \frac{\pi}{11} \right)^2 - 11 \cos^2 \frac{3\pi}{11} \\
 = & 12 \sin^2 \frac{3\pi}{11} + 4 \sin^2 \frac{5\pi}{11} + 4 \sin^2 \frac{\pi}{11} \\
 & + 4 \sin \frac{3\pi}{11} \sin \frac{5\pi}{11} - 4 \sin \frac{3\pi}{11} \sin \frac{\pi}{11} - 8 \sin \frac{5\pi}{11} \sin \frac{\pi}{11} - 11 \\
 = & 6 \left( 1 - \cos \frac{6\pi}{11} \right) + 2 \left( 1 - \cos \frac{10\pi}{11} \right) + 2 \left( 1 - \cos \frac{2\pi}{11} \right) \\
 & + 2 \left( \cos \frac{2\pi}{11} - \cos \frac{8\pi}{11} \right) - 2 \left( \cos \frac{2\pi}{11} - \cos \frac{4\pi}{11} \right) - 4 \left( \cos \frac{4\pi}{11} - \cos \frac{6\pi}{11} \right) - 11 \\
 = & -1 - 2 \left( \cos \frac{2\pi}{11} + \cos \frac{4\pi}{11} + \cos \frac{6\pi}{11} + \cos \frac{8\pi}{11} + \cos \frac{10\pi}{11} \right) \\
 = & 0,
 \end{aligned}$$

所以

$$\left( \tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} \right)^2 = 11,$$

易知  $\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} > 0$ , 故

$$\tan \frac{3\pi}{11} + 4 \sin \frac{2\pi}{11} = \sqrt{11}.$$